GNSS/INS Dynamic Model and Unscented Kalman filter

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1. Introduction

In navigation guidance and control of land, underwater or aerial vehicles, there are several coordinate systems (or frames) intensively used in design and analysis. For ease of reference, we summarize the coordinate systems adopted in our project, which include:

- the geodetic coordinate system,
- the earth-centered-earth fixed (ECEF) coordinate system,
- the local north-east-down (NED) coordinate system,
- the vehicle-carried NED coordinate system,
- the body coordinate system.

We need to point out that a land vehicle and miniature unmanned aircraft and rotorcraft are normally utilzed at low speeds and in small regions. This is crucial to some simplification made in the coordinate transforrnation (e.g., omitting unimport items in the transformation between the local NED frame and the body frame).

The GNSS/INS projects are built on the uniform mathematical platform and contain the following applications:

a) Vertical Gyro System,
b) Attitude Heading Reference System,
c) Single Antenna GNSS/INS System,
d) Dual Antenna GNSS/INS System.

The Vertical Gyro System (VGS) application provides dynamic roll, pitch and yaw data. The roll and pitch values are stabilized by using the 3-axis accelerometer during the motionless period of time. During a motion the tilt angles are stabilized applying the second order complementary filter. The Vertical Gyro System can also output a free integrated yaw angle value that is not stabilized by magnetometer measurements.

The Attitude Heading Reference System (AHRS) application includes additional 3-axis magnetometer measurements. This enables the computation of dynamic yaw as well as dynamic roll and pitch data. The dynamic yaw values are stabilized using the 3-axis magnetometer measurements. The roll and pitch values are stabilized by using the 3-axis accelerometer measurements during the motionless period of time and the complementary filter during a motion.

The Single Antenna GNSS/INS provides the calculation of dynamic attitude angles as well as navigation coordinates and velocities. The roll, pitch and velocity values are stabilized by using the 3-axis accelerometer measurements and the zero-velocity updating during the motionless period of time. The dynamic attitude angles and navigation coordinates and velocity are stabilized using GNSS position and velocity nonzero measurements and 3-axis magnetometer measurements.

The Dual Antenna GNSS/INS provides the calculation of dynamic attitude angles and navigation coordinates and velocities. The roll, pitch, position and velocity values are stabilized by using the 3-axis accelerometer measurements and the zero-velocity updating during the motionless period of time. The dynamic attitude angles and navigation coordinates and velocities are stabilized using GNSS position and nonzero velocity measurements and GNSS true heading measurements.

2. Coordinate systems

The geodetic coordinate system (see Fig. 1) is widely used in GNSS-based navigation. We note that it is not a usual Cartesian coordinate system but a system that characterizes a coordinate point near the earth's surface in terms of latitude, longitude and height, which are respectively denote by φ, λ and *h*. The latitude measures the angle (ranging from -90° to 90°) between the equatorial plane and normal of the reference ellipsoid that passes through the measured point. The longitude measures the rotational angle (ranging from -180° to 180°) between the Prime Meridian and the measured point. The height is the local vertical distance between the measured point and reference ellipsoid.

Important parsmeters assoiated with the geodetic frame include

- the semi-major axis R_a,
- the flattering factor *f*,
- the semi-minor axis R_b,
- the first eccentricity *e*,
- the prime vertical radius of curvature *N*.

These parameters are either defined (items 1 and 2) or derived (items 3 to 5) based on the WGS 84 (word geodetic system 84, which was originally proposed in 1984 and lastly updated in 2004) ellipsoid model. More specifically, we have

$$R_a = 6{,}378{,}137.0m,$$

$$f = \frac{1}{298.257223563},$$

$$R_b = R_a(1-f),$$

$$e = \frac{\sqrt{R_a^2 - R_b^2}}{R_a},$$

$$N = \frac{R_a}{\sqrt{1-e^2 sin^2 \varphi}}.$$

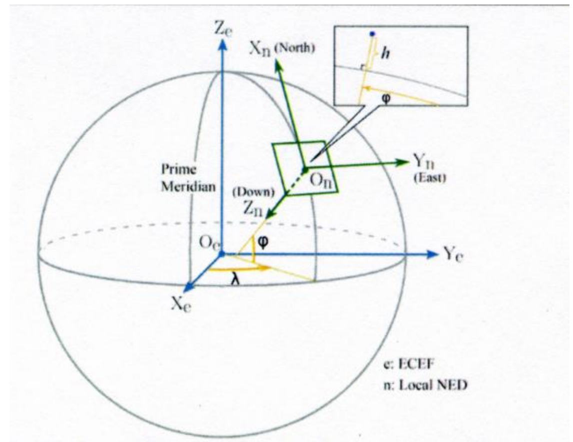

Fig. 1. Geodetic, ECEF and local NED coordinate systems

The ECEF coordinate system rotates with earth around its spin axis. As such, a fixed point on the earth surface has a fixed set of coordinates. The origin and axes of the ECEF coordinate system (see Fig. 1) are defined as follows:

- The origin (denoted by O_e) is located at the center of the earth.
- The Z-axis (denoted by Z_e) is along the spin axis of the earth, pointing to the north pole.
- The X-axis (denoted by X_e) intersects the sphere of the earth at 0° latitude and 0° longitude.
- The Y-axis (denoted by Y_e) is orthogonal to the Z- and X-axes with the usual right-hand rule.

The local NED coordinate system is also known as a navigation or ground coordinate system. It is a coordinate frame fixed to the earth's surface. Based on the WGS 84 ellipsoid model, its origin and axes are defined as the following (see Fig.1):

- The origin (denoted by O_n *or* O_o) is arbitrarily fixed to a point on the earth's surface.
- The X-axis (denoted by X_n) points toward the ellipsoid north (geodetic north).

- The Y-axis (denoted by Y_n) points toward the ellipsoid east (geodetic east).
- The Z-axis (denoted by Z_n) points downward along the ellipsoid normal.

The local NED frame plays a very important role in flight control and navigation. Navigation of land and unmanned aerial vehicles are normally carried out within this frame.

We also note that in our work, we normally select the launching point, which is also the sensor initialization point, in each ground and flight test as the origin of the local NED frame.

The vehicle-carried NED system is associated with the moving vehicle. Its origin and axes are given by the following:

- The origin is located at the center of gravity of the moving vehicle.
- The X-axis points toward the ellipsoid north (geodetic north).
- The Y-axis points toward the ellipsoid east (geodetic east).
- The Z-axis points downward along ellipsoid normal.

Strictly speaking, the axis directions of the vehicle-carried NED frame vary with respect to the vehicle movement and are thus not aligned with those of the local NED frame. However, as mentioned earlier, the land vehicle and miniature unmanned aircraft and rotorcraft move only in a small region with low speed, which results in the directional difference being completely neglectable. As such, it is reasonable to assume that the directions of the vehicle-carried and local NED coordinate systems constantly coincide with each other.

The body coordinate system is vehicle-carried and is directly defined on the body of the moving vehicle. Its origin and axes are given by the following:

- The origin is located at the center of gravity of the moving vehicle.
- The X-axis points forward, lying in the symmetric plane of the moving vehicle.
- The Y-axis is starboard (the right side of the vehicle).
- The Z-axis points downward to comply with the right-side rule.

3. <u>Dynamic equations in local NED coordinate system</u>

Let us introduce the following mathematical objects (see Fig. 2):

O_o is the origin of local NED coordinate system,

$\boldsymbol{G}$ is the transform matrix between GEI (geocentric equatorial inertial system) and ECEF coordinate system,

$\boldsymbol{GR_o}$ is the radius-vector of the origin O_o,

$\boldsymbol{R}$ is the radius-vector of moving point M (the origin of body-frame coordinate system),

$\boldsymbol{r} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Is the radius vector of point M in the local NED coordinate system,

$\boldsymbol{L_o}$ is the transform matrix between ECEF coordinate system and local NED coordinate system,

$\boldsymbol{R_a}$ is the radius vector of point A (GNSS antenna location),

$\boldsymbol{GL_o r}$ is the radius vector of point M in the GEI coordinate system,

$\boldsymbol{\rho_a}$ is the radius vector of point A in the body-fixed frame,

$\boldsymbol{B}$ is the transform matrix between local NED coordinate system and body-fixed frame,

$\boldsymbol{GL_o B \rho_a}$ is the radius-vector of point A in the GEI coordinate system.

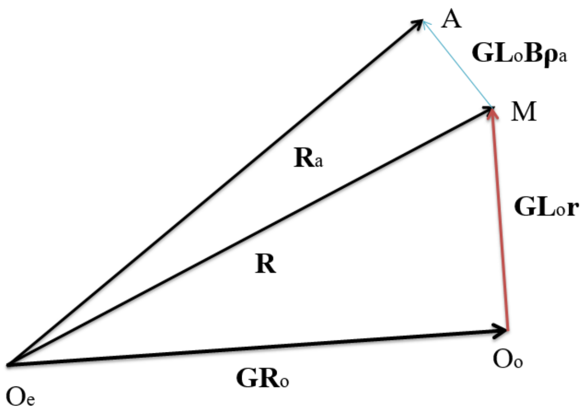

Fig. 2. Designations for origins of coordinate systems and radius-vectors.

In GEI coordinate system we can write the vector-matrix expressions

$$\boldsymbol{R} = \boldsymbol{GR_o} + \boldsymbol{GL_o r}$$

$$\frac{d\boldsymbol{R}}{dt} = \frac{d\boldsymbol{G}}{dt}\boldsymbol{R_o} + \frac{d\boldsymbol{G}}{dt}\boldsymbol{L_o r} + \boldsymbol{GL_o v}$$

$$\frac{d^2\boldsymbol{R}}{dt^2} = \frac{d^2\boldsymbol{G}}{dt^2}\boldsymbol{R_o} + \frac{d^2\boldsymbol{G}}{dt^2}\boldsymbol{L_o r} + 2\frac{d\boldsymbol{G}}{dt}\boldsymbol{L_o v} + \boldsymbol{GL_o}\frac{d\boldsymbol{v}}{d\boldsymbol{t}}$$

Where:

$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is the velocity vector of point M in the local NED coordinate system.

It follows from equations defined above that the kinematical differential equations for the origin of body-fixed frame A are given by

$$\frac{dx_1}{dt} = v_1$$

$$\frac{dx_2}{dt} = v_2$$

$$\frac{dx_3}{dt} = v_3$$

$$\frac{dv_1}{dt} = \Omega_e^2 sin^2\varphi_o x_1 + \Omega_e^2 sin\varphi_o cos\varphi_o x_3 - 2\Omega_e sin\varphi_o v_2 + B_{11}f_x + B_{12}f_y + B_{13}f_z$$

$$\frac{dv_2}{dt} = \Omega_e^2 x_2 + 2\Omega_e sin\varphi_o v_1 + 2\Omega_e cos\varphi_o v_3 + B_{21}f_x + B_{22}f_y + B_{23}f_z$$

$$\frac{dv_3}{dt} = \Omega_e^2 sin\varphi_o cos\varphi_o x_1 + \Omega_e^2 cos^2\varphi_o x_3 - 2\Omega_e cos\varphi_o v_2 + B_{31}f_x + B_{32}f_y + B_{33}f_z + g_o$$

Where:

Ω_e is the earth angular speed

(7.292115e-5 $\frac{rad}{sec}$),

g_o is the earth gravitational constant (9.780327$\frac{m}{sec^2}$),

φ_o is the latitude of the origin O_o,

f_x, f_y, f_z are the components of specific force relative to the body-fixed frame.

The matrix differential equation for the transform matrix $\boldsymbol{B}$ is defined as

$$\frac{d\boldsymbol{B}}{dt} = \boldsymbol{B\Omega} - \boldsymbol{\Lambda_o B}$$

Where anti-symmetric matrices $\boldsymbol{\Omega}$ and $\boldsymbol{\Lambda_o}$ are equal to

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$\boldsymbol{\Lambda_o} = \begin{bmatrix} 0 & \Omega_e sin\varphi_o & 0 \\ -\Omega_e sin\varphi_o & 0 & -\Omega_e cos\varphi_o \\ 0 & \Omega_e cos\varphi_o & 0 \end{bmatrix}$$

and $\omega_x, \omega_y, \omega_z$ are the components of absolute angular velocity relative to the body-fixed frame.

From the above matrix equation we can obtain the kinematic differential equations for standard yaw, pitch and roll (ψ, θ, φ) angles (see Fig. 3)

$$\frac{d\varphi}{dt} = \omega_x + tan\theta sin\varphi\omega_y + tan\theta cos\varphi\omega_z - \Omega_e cos\varphi_o \frac{cos\psi}{sin\theta}$$

$$\frac{d\theta}{dt} = cos\varphi\omega_y - sin\varphi\omega_z + \Omega_e cos\varphi_o sin\psi$$

$$\frac{d\psi}{dt} = \frac{sin\varphi}{cos\theta}\omega_y + \frac{cos\varphi}{cos\theta}\omega_z + \Omega_e sin\varphi_o - \Omega_e cos\varphi_o tan\theta cos\psi$$

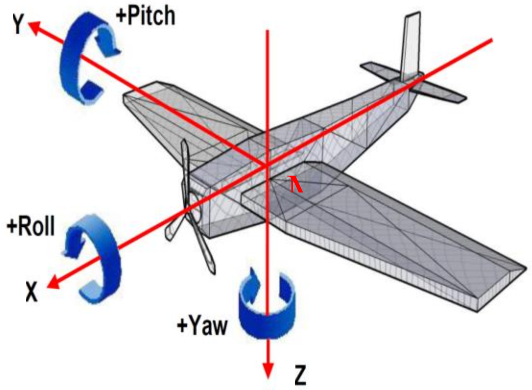

Fig. 3. Body-fixed frame and yaw, pitch and roll Euler angles.

The transform matrix $\boldsymbol{B}$ in terms of the Euler angles is expressed as

$$B = \begin{bmatrix} cos\theta cos\psi & sin\varphi sin\theta cos\psi - cos\varphi sin\psi & cos\varphi sin\theta cos\psi + sin\varphi sin\psi \\ cos\theta sin\psi & sin\varphi sin\theta sin\psi + cos\varphi cos\psi & cos\varphi sin\theta sin\psi - sin\varphi cos\psi \\ -sin\theta & sin\varphi cos\theta & cos\varphi cos\theta \end{bmatrix}$$

4. Unscented Kalman filter with nonlinear dynamic and measuring processes

The unscented Kalman filtration is a nonlinear version of Kalman filtration, which deals with the case governed by the nonlinear stochastic dynamic system in the discrete time form

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}(\mathbf{x}_k, \mathbf{w}_k) \\ \mathbf{z}_k &= \mathbf{g}(\mathbf{x}_k, \mathbf{e}_k) \end{aligned} \tag{1}$$

Where

$\mathbf{x}_k$ is the dynamic state vector having the mean $\hat{\mathbf{x}}_k$ and covariance $\mathbf{P}_k$,

$\mathbf{w}_k$ is the process noise vector,

$\mathbf{z}_k$ is the measurement vector,

$\mathbf{e}_k$ is the measurement noise vector.

In system (1), both vectors $\mathbf{w}_k$ and $\mathbf{e}_k$ are zero mean Gaussian white sequences. Because a white sequence is a sequence of a zero-mean random variable that is uncorrelated time-wise, the covariance matrix associated with $\mathbf{w}_k$ is given as

$$E\left[\mathbf{w}_k \mathbf{w}_m^T\right] = \begin{cases} \mathbf{Q}_k, & m = k \\ \mathbf{0}, & m \neq k \end{cases}. \tag{2}$$

The measurement covariance matrix is written as

$$E\left[\mathbf{e}_k \mathbf{e}_m^T\right] = \begin{cases} \mathbf{R}_k, & m = k \\ \mathbf{0}, & m \neq k \end{cases}. \tag{3}$$

The system noise $\mathbf{w}_k$ and measurement noise $\mathbf{e}_k$ are assumed to be uncorrelated, i.e.

$$E\left[\mathbf{w}_k \mathbf{e}_m^T\right] = 0 \quad for\, all\; m, k. \tag{4}$$

In the unscented Kalman filter (UKF), system noises are generated and are passed through the system process model. The state vector $\mathbf{x}_k$ and the system noise $\mathbf{w}_k$ are augmented to create the augmented state vector $\mathbf{s}_k$:

$$\mathbf{s}_k = \begin{bmatrix} \mathbf{x}_k \\ \mathbf{w}_k \end{bmatrix}. \tag{5}$$

Similarly, the state vector $\mathbf{x}_k$ and the measurement noise $\mathbf{e}_k$ are augmented to create the augmented state vector $\mathbf{r}_k$:

$$\mathbf{r}_k = \begin{bmatrix} \mathbf{x}_k \\ \mathbf{e}_k \end{bmatrix}. \tag{6}$$

The UKF is based on transforming a set of deterministically chosen points, called the sigma points, through the nonlinear system process and measurement models. The unscented transformation refers to the procedure of obtaining a set of the sigma points from the given mean and covariance [1]. The implementation of the UKF is often substantially easier and does not require to derive analytic partial derivatives as in the extended Kalman filter.

The estimated state vector and covariance are augmented with the mean and covariance of the process noise:

$$\hat{\mathbf{s}}_k = \begin{bmatrix} \hat{\mathbf{x}}_k \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{P}_k^a = \begin{bmatrix} \mathbf{P}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_k \end{bmatrix}. \tag{7}$$

A set of *2l+1* sigma points is derived from the augmented state and covariance where *l* is the dimension of the augmented state

$$\begin{aligned}\boldsymbol{\chi}_k^0 &= \hat{\mathbf{s}}_k \\ \boldsymbol{\chi}_k^i &= \hat{\mathbf{s}}_k + \left(\sqrt{(l+\lambda)\mathbf{P}_k^a}\right)_i, \; i = 1,\ldots,l \\ \boldsymbol{\chi}_k^i &= \hat{\mathbf{s}}_k - \left(\sqrt{(l+\lambda)\mathbf{P}_k^a}\right)_i, \; i = l+1,\ldots,2l\end{aligned} \tag{8}$$

Where $\left(\sqrt{(l+\lambda)\mathbf{P}_k^a}\right)_i$ is the *i-th* column of the matrix square root of $(l+\lambda)\mathbf{P}_k^a$; $\lambda = \alpha^2(l+\kappa) - l$ is a scaling factor. The constant α determines the spread of the sigma points around $\hat{\mathbf{s}}_k$; it is set a small positive value, typically in the range $10^{-4} < \alpha < 1$. The constant κ is a secondary scaling factor that is usually set to *0*.

The sigma points (8) are propagated through the nonlinear system process model (1)

$$\mathbf{x}_{k+1}^i = \mathbf{f}\left(\boldsymbol{\chi}_k^i\right), \; i = 0,\ldots,2l \tag{9}$$

The weighted sigma points are recombined to produce the predicted state

$$\hat{\mathbf{x}}_{k+1}^- = \sum_{i=0}^{2l} c_m^i \mathbf{x}_{k+1}^i \tag{10}$$

and predicted covariance

$$\mathbf{P}_{k+1}^- = \sum_{i=0}^{2l} c_c^i \left(\mathbf{x}_{k+1}^i - \hat{\mathbf{x}}_{k+1}^-\right)\left(\mathbf{x}_{k+1}^i - \hat{\mathbf{x}}_{k+1}^-\right)^T \tag{11}$$

Where the weights for state and covariance are given by:

$$\begin{aligned}c_m^0 &= \frac{\lambda}{l+\lambda} \\ c_c^0 &= \frac{\lambda}{l+\lambda} + \left(1 - \alpha^2 + \beta\right) \\ c_m^i &= c_c^i = \frac{1}{2(l+\lambda)}, \; i = 1,\ldots,2l\end{aligned} \tag{12}$$

In the above equations, the subscripts *m* and *c* refer to the mean and covariance, respectively. β is used to take account of prior knowledge on the distribution of the state vector, and $\beta = 2$ is the optimal choice for Gaussian distributions.

The predicted state vector (10) and covariance (11) are augmented as before, except at present with the mean and covariance of the measurement noise

$$\hat{\mathbf{s}}_{k+1}^- = \begin{bmatrix}\hat{\mathbf{x}}_{k+1}^- \\ \mathbf{0}\end{bmatrix}, \qquad \mathbf{P}_{k+1}^a = \begin{bmatrix}\mathbf{P}_{k+1}^- & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_k\end{bmatrix}. \tag{13}$$

As before, a set of *2n+1* sigma points is derived from the augmented state and covariance where *n* is the dimension of the augmented state

$$\begin{aligned}\boldsymbol{\pi}_{k+1}^0 &= \hat{\mathbf{s}}_{k+1}^- \\ \boldsymbol{\pi}_{k+1}^i &= \hat{\mathbf{s}}_{k+1} + \left(\sqrt{(n+\mu)\mathbf{P}_{k+1}^a}\right)_i, \; i = 1,\ldots,n \\ \boldsymbol{\pi}_{k+1}^i &= \hat{\mathbf{s}}_{k+1} - \left(\sqrt{(n+\mu)\mathbf{P}_{k+1}^a}\right)_i, \; i = l+1,\ldots,2n\end{aligned} \tag{14}$$

where

$\left(\sqrt{(n+\mu)\mathbf{P}_{k+1}^a}\right)_i$ is the *i-th* column of the matrix square root of $(n+\mu)\mathbf{P}_{k+1}^a$;

$\mu = \alpha^2(n+\kappa) - n$.

During the update state, the sigma points are transformed through the measurement model

$$\mathbf{z}_{k+1}^i = \mathbf{g}\left(\boldsymbol{\pi}_{k+1}^i\right). \tag{15}$$

The predicted measurement is computed as

$$\hat{\mathbf{z}}_{k+1}^- = \sum_{i=0}^{2n} b_m^i \mathbf{z}_{k+1}^i \tag{16}$$

and predicted measurement covariance as

$$\mathbf{M}_{k+1} = \sum_{i=0}^{2n} b_c^i \left(\mathbf{z}_{k+1}^i - \hat{\mathbf{z}}_{k+1}^-\right)\left(\mathbf{z}_{k+1}^i - \hat{\mathbf{z}}_{k+1}^-\right)^T \tag{17}$$

where the weights for state and covariance are given by

$$\begin{aligned}b_m^0 &= \frac{\mu}{n+\mu} \\ b_c^0 &= \frac{\mu}{n+\mu} + \left(1 - \alpha^2 + \beta\right) \\ b_m^i &= b_c^i = \frac{1}{2(n+\mu)}, \; i = 1,\ldots,2n\end{aligned} \tag{18}$$

The state and measurement cross-covariance matrix

$$\mathbf{N}_{k+1} = \sum_{i=0}^{2n} b_c^i \left(\mathbf{y}_{k+1}^i - \hat{\mathbf{x}}_{k+1}^-\right)\left(\mathbf{z}_{k+1}^i - \hat{\mathbf{z}}_{k+1}^-\right)^T \tag{19}$$

is used to compute the UKF gain

$$\mathbf{K}_{k+1} = \mathbf{N}_{k+1} \cdot \mathbf{M}_{k+1}^{-1} \tag{20}$$

Where $\mathbf{y}_{k+1}^i$ denotes the block of the augmented vector $\boldsymbol{\pi}_{k+1}^i$ appropriate to the dynamic state vector.

As with the Kalman filter, the updated state is the predicted state plus the innovation weighted by the Kalman gain

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{k+1}^- + \mathbf{K}_{k+1}\left(\mathbf{z}_{k+1} - \hat{\mathbf{z}}_{k+1}^-\right). \tag{21}$$

The updated covariance is the predicted covariance minus the predicted measurement covariance weighted by the Kalman gain

$$\mathbf{P}_{k+1} = \mathbf{P}_{k+1}^{-} - \mathbf{K}_{k+1} \cdot \mathbf{M}_{k+1} \cdot \mathbf{K}_{k+1}^{T} \cdot \quad (22)$$

5. References

1. Rudolph van der Merwe, Eric A. Wan, Simon I. Julier "Sigma-Point Kalman Filters for Nonlinear Estimation and Sensor-Fusion: Applications to Integrated Navigation", AIAA Guidance, Navigation, and Control Conference and Exhibit (16 August 2004).